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AFDELING TOEGEPASTE WISKUNDE (DEPARTMENT OF APPLIED MATHEMATICS) TW 222/81 NOVEMBER

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A SEMI INFINITE TRUMPET-SHAPED MODEL OF THE COCHLEA

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Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification : 35J25, 76ZXX, 30C20, 30E15

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by

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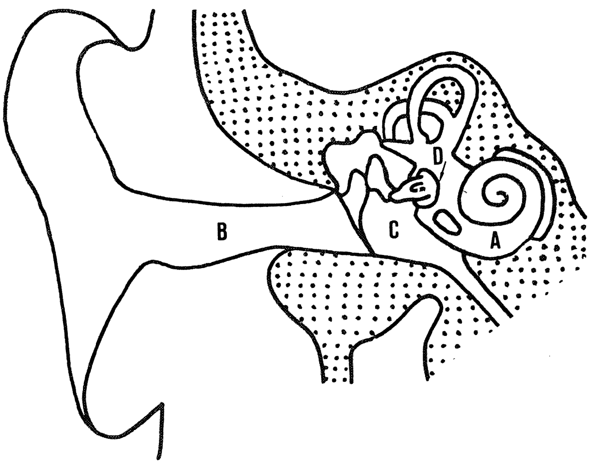
ABSTRACT

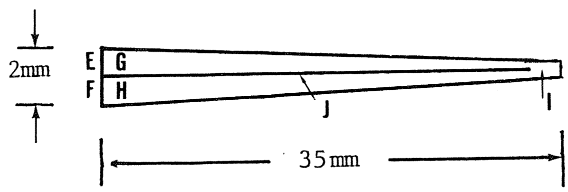
In this paper a semi infinite trumpet-shaped model of the cochlea is presented. By means of conformal mapping and other general techniques of complex variables an asymptotic solution is obtained. The result of this paper meets the well-known effects as appear in other mathematical models on the subject. Finally an asymptotic solution is given as the radius of the trumpet vanishes.

KEY WORDS & PHRASES: *elliptic partial differential equation, semi infinite cochlea model, theory of complex variables, biomathematics*

1. INTRODUCTION

The cochlea, part of the inner ear, is a small fluid-filled snail-shaped chamber which contains the biological structures that convert mechanical acoustic signals into neural signals. The essential role of the outer and middle ear appears to be that of an impedance matching device which transduces airborne acoustic energy into motion of the perilymphatic fluid contained within the cochlea (Fig. 1.1.) To achieve some understanding about the fluid mechanics in the inner ear we uncoil the cochlea. This tube consists, roughly spoken, of two equal parts: the scala vestibuli and the scala tympani. Both parts are separated by the basilar membrane, where the mechanical signals are converted into neural activity. At the basic end we find two other membranes, one which passes the stimuli form the middle ear into the scala vestibuli (the oval window); the other, at the end of the scala tympani, is called the round window. The basilar membrane ends just short of the cochlear apex leaving a small opening : the helicotrema (Fig. 1.2.)





A Cochlea, B Outer ear, C Middle ear
D Stirrup, E Oval window F Round window.
G Scala vestibuli, H Scala tympani,
I Helicotrema, J Basilar membrane.

Fig. 1.1. The auditory system   Fig. 1.2. Uncoiled cochlea

The airborne acoustic signal sets the eardrum into motion which in turn causes motion of the middle ear bones. One of these bones (the stirrup) is attached to the oval window. The oscillatory motion of the oval window is propagated to the cochlear fluid which in turn causes the round window to oscillate. This fluid also forces the basilar membrane to oscillate in a

specific way. It is the motion of this membrane that has our interest in this note.
We shall consider a two-dimensional mathematical model, hence we neglect physical properties that do depend on the width of the cochlea. The properties of the cochlear fluid are described by the local velocity vector $(n(u,v,t), m(u,v,t))$, the pressure $p(u,v,t)$ and the fluid density ρ. The linearised equations of motion are

$$(1.1.a) \qquad \rho \frac{\partial n}{\partial t} = - \frac{\partial p}{\partial u} ,$$

$$(1.1.b) \qquad \rho \frac{\partial m}{\partial t} = - \frac{\partial p}{\partial v} .$$

If the cochlear fluid is taken as incompressible, the equation of continuity reads

$$(1.2) \qquad \frac{\partial n}{\partial u} + \frac{\partial m}{\partial v} = 0 .$$

Conditions (1.1) and (1.2) make that p satisfies the potential equation $\Delta p = 0$. Since there exists geometrical symmetry, i.e. $p(u,-v,t) = - p(u,v,t)$, we neglect the scala tympani.

The boundary condition at the oval window may be taken as $p = p_0 \exp(i\omega t)$ to express the harmonic character of the input. Because of the linearity of the model p_0 may be given by any arbitrary value. That is, we consider an oscillatory process where all time dependent variables contain the same factor $\exp(i\omega t)$. For the sake of convenience we omit this factor in the present paper, moreover we set $p_0 = 1$.

The boundary condition at the basilar membrane describes the reaction of this membrane upon the external pressure force exerted at it. It will depend on mass, damping and stiffness of the membrane per unit area. Experiences of DE BOER [2] learned that

$$(1.3) \qquad p = A(u) \frac{\partial p}{\partial v} , \qquad A(u) = \frac{1}{2\rho} \left(0.05 - \frac{10^9 e^{-3u}}{\omega^2} - i \frac{100}{\omega}\right) ,$$

where the upperbound of the frequency ω is about 16 kHz and the lower bound 2 kHz. The density ρ is taken 1. Equation (1.3) is the so-called basilar membrane condition. A remarkable feature is that the real part of the function $A(u)$ vanishes for a positive value of u. This "point of resonance" will have our special interest. The other conditions are obvious.

2. SEMI-INFINITE "TRUMPET-SHAPED" MODEL

Consider the following model of the cochlea as a semi infinite trumpet-shaped strip (see Fig. 2.1). The curvature in this model is quite arbitrary but should be choosen such that there exists a comformal mapping which transforms the trumpet-shaped model into a infinite strip.

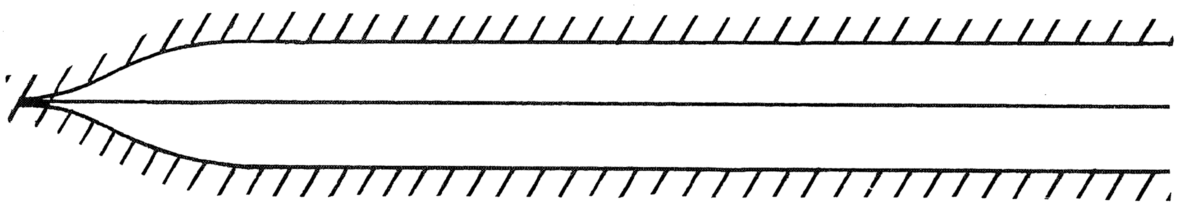


Fig. 2.1. Trumpet-shaped model

The upper and lower curve in Fig. 2.1 represent the bony housing of the cochlea, the straight line is the basilar membrane. Intersection of the upper curve with the membrane represents the oval window, intersection of the lower curve the round window. Finally the model represents at infinity the helicotrema.

In the present paper we choose as a conformal mapping the Joukowski transformation, i.e.

(2.1) $$f(w) = u_3 \left[w - \frac{u_1}{w} + u_2\right],$$

where $w = u + iv$, $f(w) = x + iy$ and $u_1, u_2, u_3 \in \mathbb{R}$. Of main interest is the point where the real part of $A(u)$ vanishes, that is $u = u_0 = \frac{1}{3} \ln \frac{2.10^{10}}{\omega^2}$ (see 1.3). Furthermore we wish to obtain that $f(u_0) = 0$ or

(2.2) $$u_0 - \frac{u_1}{u_0} - u_2 = 0,$$

(for u_3 see (2.4)), which leaves one degree of freedom. With this we are able to regulate the slope of the curve.

Using the Jacobian of the transformation at v = 0 we obtain for the basilar membrane condition $B(x)\,\frac{\partial p}{\partial y} = p$ the following

$$B(x) = A(u)\left(u_3 + \frac{u_1}{u^2}\right), \quad u = \tfrac{1}{2}\left(-u_2 + \frac{x}{u_3} + \left[u_2^2 + \frac{x^2}{u_3^2} - \frac{2u_2x}{u_3} + 4u_1\right]^{1/2}\right).$$

With (2.2) and for $|x|$ small u can be approximated by

$$u \sim \frac{x}{2u_3} + u_0 .$$

This yields

$$\text{(2.3)} \qquad B(x) \sim \left(0.025\left(1 - i\,\frac{2000}{\omega}\right) + \frac{10^9}{\omega^2}\exp\left(-\frac{3x}{2u_3} - 3u_0\right)\right) \times \left[u_3 + u_1 / \left(\frac{x}{2u_3} - u_0\right)^2\right] .$$

Now set

$$\text{(2.4)} \qquad u_3 = \frac{2}{3} \text{ and } \mu^{-1} = 0.025\left(u_3 + \frac{u_1}{u_0^2}\right)\left(1 + \frac{4.10^6}{\omega^2}\right)^{1/2}$$

then (2.3) reads

$$B(x) \sim \frac{1}{\mu}\left(e^{i\varepsilon} + e^{-x+\nu}\right),$$

where $\varepsilon = -\arctan\left(\frac{4.10^6\omega^2 + 16.10^{12}}{\omega^4 + 4.10^6\omega^2}\right)^{1/2}$ and $\nu = -\tfrac{1}{2}\ln\left(1 + \frac{4.10^6}{\omega^2}\right)$.

The constant $\nu(\omega)$ causes a shift of the curve. It is obvious that for increasing ω the curve moves to the left. Apart from this important phenomenon we neglect ν (set $\nu=0$).

In Fig. 2.2 respectively Fig. 2.3 the model before and after conformal mapping is shown.

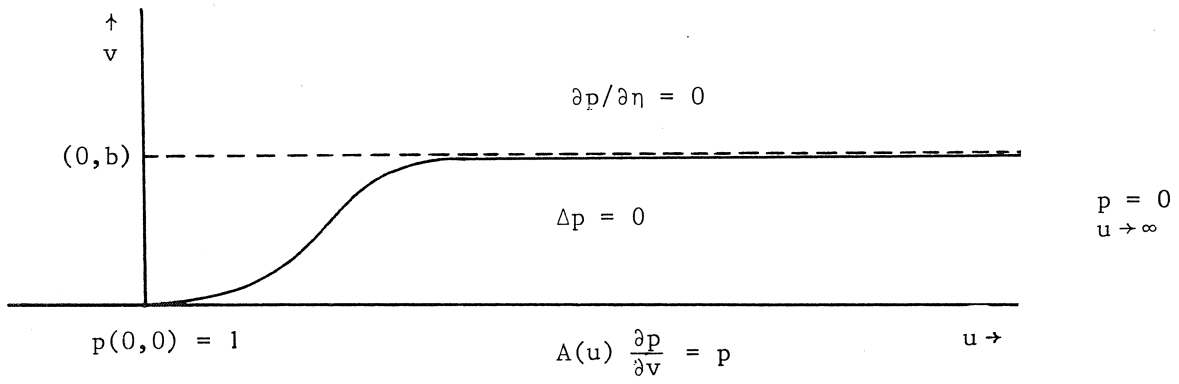


Fig. 2.2. Original model, η is the normal vector to the curve.

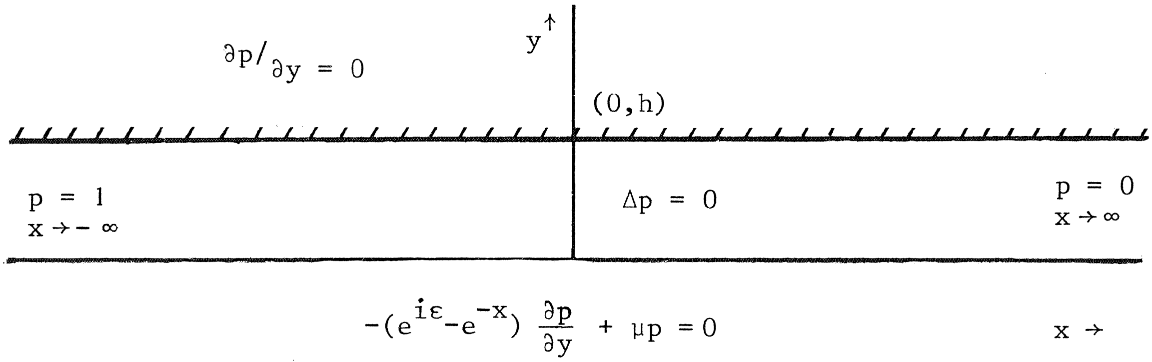


Fig. 2.3. Transformed model , $h = \frac{2}{3} b$.

3. INFINITE STRIP MODEL

In this section we build up a solution for the above mentioned model. Main tool will be the residue theorem. We rewrite the basilar membrane condition as

$$(3.1) \qquad (1-e^{-(x+i\varepsilon)}) \frac{\partial p}{\partial y} - \mu e^{-i\varepsilon} p = 0 .$$

Let $\mu e^{-i\varepsilon} = \delta$. We propose the following integral representation

$$(3.2) \qquad p(x,y) = 1 - \frac{1}{2\pi i} \int_L e^{s(x+i\varepsilon)} \frac{\cos s(h-y)}{\cos sh} \phi(s)\, ds .$$

To achieve some understanding about the behaviour of p as $|x| \to \infty$ we consider the auxiliar problem (3.3)

$$(3.3)\qquad \begin{cases} \Delta q = 0 \text{ on the upper half plane } y > 0, \\ (1-e^{-(x+i\varepsilon)})\ \dfrac{\partial q}{\partial y} - \delta q = 0 \quad \text{if } y = 0, \\ q = 1 \quad \text{if } x \to -\infty\ ,\ q = 0 \quad \text{if } x \to \infty\ . \end{cases}$$

Here the solution is $q(z) = (1-e^{z+i\varepsilon})^{-i\delta}$, where $z = x+iy$. We introduce a similar representation as in (3.2) for q, that is

$$(3.4)\qquad (1-e^{x+i\varepsilon})^{-i\delta} = 1 - \frac{1}{2\pi i} \int_L e^{s(x+i\varepsilon)}\ \psi(s)\, ds\ ,\ y = 0\ .$$

Set $x + i\varepsilon = t$ and apply the Laplace transform

$$\psi(s) = \int_{-\infty}^{\infty} e^{-st}\,[1-(1-e^{t})^{-i\delta}]\, dt\ ,$$

or with $\tau = e^{t}$

$$(3.5)\qquad \psi(s) = \int_0^{\infty} \tau^{-s-1}\,[1-(1-\tau)^{-i\delta}]\, d\tau\ .$$

Conditions for $\psi(s)$ to be regular are

$$(3.6a)\qquad \int_0^1 \tau^{-s-1}\,[1-(1-\tau)^{-i\delta}]\, d\tau \text{ convergent as } \tau \to 0\ ,$$

$$(3.6b)\qquad \int_1^{\infty} \tau^{-s-1}\,[1-(1-\tau)^{-i\delta}]\, d\tau \text{ convergent as } \tau \to \infty\ .$$

It is obvious that (3.6a) yields $\mathrm{Re}(s) < 1$. Since $\mathrm{Re}(-i\delta) < 0$ for the inner ear case the part between brackets in (3.6b) can be approximated by 1 and hence we find that $\psi(s)$ is regular on the strip $0 < \mathrm{Re}(s) < 1$. A proper choice of L is $\{c+i\sigma \mid 0 < c < 1\ ,\ -\infty < \sigma < \infty\}$. Furthermore we can express (3.5) as

$$(3.7)\qquad \psi(s) = \frac{\Gamma(1-s)\,\Gamma(1-i\delta)}{s\,\Gamma(1-s-i\delta)} - (-1)^{-i\delta}\,\frac{\Gamma(s+i\delta)\Gamma(1-i\delta)}{\Gamma(s+1)}\ .$$

The boundary value of the auxiliar problem (3.3) as $x \to \infty$ immediately follows from the representation (3.4), since on L we have $\text{Re}(sx) \to \infty$ as $x \to -\infty$. The boundary value at $x = +\infty$ is less obvious. It is obtained by shifting L across the pole of ψ at $s = 0$. Taking into account the residue at this pole we infer that indeed q vanishes as $x \to \infty$.

For ϕ introduced in (3.2) we expect the same phenomenon as $x \to \infty$: we suppose that $\phi(s)$ behaves like $\psi(s)$ so that $\phi(s)$ has a pole at $\text{Re}(s) = 0$

With this assumption we find for the boundary condition

$$\lim_{x\to\infty} p(x,0) = 1 - \frac{1}{2\pi i} \lim_{x\to\infty} \int_{c-i\infty}^{c+i\infty} e^{s(x+i\varepsilon)} \frac{\cos s(h-y)}{\cos sh} \phi(s)\, ds$$

$$= 1 - \frac{1}{2\pi i} \lim_{x\to\infty} \int_{c-1-i\infty}^{c-1+i\infty} e^{s(x+i\varepsilon)} \frac{\cos s(h-y)}{\cos sh} \phi(s) ds - \lim_{s\to 0} s\,\phi(s)$$

$$= 1 - \lim_{s\to 0} s\,\phi(s) = 1 - \lim_{s\to 0} s\,\psi(s) = 0,$$

where we assumed that $\lim_{|\text{Im } s|\to\infty} \phi(s) = 0$ (proof later). This leaves only one boundary condition that should be satisfied, namely the basilar membrane condition (3.1). With the aid of (3.2) we can rewrite (3.1) as

$$(3.8a) \quad \int_{c-i\infty}^{c+i\infty} e^{s(x+i\varepsilon)} (\delta - s\tan sh)\, \phi(s) + e^{(s-1)(x+i\varepsilon)} s\tan sh\, \phi(s) ds = 2\pi i\delta$$

or

$$(3.8b) \quad \int_{c-i\infty}^{c+i\infty} e^{s(x+i\varepsilon)} (\delta - s\tan sh)\phi(s) ds + \int_{c-1-i\infty}^{c-1+i\infty} e^{s(x+i\varepsilon)} (s+1)\tan(s+1)h\, \phi(s+1) ds = 2\pi i\delta$$

Now we shift the contour of the second integral to the contour of the first integral of (3.8b). Here we pass a pole at $\text{Re}(s) = 0$ of the function $\phi(s+1)$. We prescribe the residue of the function $e^{s(x+i\varepsilon)}(s+1) \tan (s+1)\, h\, \phi(s+1)$ at this pole to have the value δ. Existence of this pole is proved later. Hence we find for (3.8b)

$$\int_{c-i\infty}^{c+i\infty} e^{s(x+i\varepsilon)}\,[(\delta - s\tan sh)\,\phi(s) + (s+1)\tan(s+1)h\,\phi(s+1)]\,ds = 0$$

for all $x \in \mathbb{R}$, so

(3.9) $$\frac{\phi(s+1)}{\phi(s)} = \frac{s\tan sh - \delta}{(s+1)\tan(s+1)h}\ .$$

The problem is solved when we can obtain $\phi(s)$ from the functional equation (3.9) and when this $\phi(s)$ meets the conditions imposed. We are not able to solve (3.9) explicitly,and we proceed with the assumption that h is small. In that event it is plausible to replace (3.9) by

(3.10) $$\frac{\phi(s+1)}{\phi(s)} = \frac{s^2-k^2}{(s+1)^2}\ ,\quad k^2=\delta/h.$$

A particular function which satisfies this homogeneous functional equation is

$$\phi^*(s) = \frac{\Gamma(s+k)\,\Gamma(-s)}{\Gamma(1+k-s)\Gamma(s+1)}\ ,$$

where Γ is the gamma function.

<u>LEMMA 3.1</u>. *If* $\phi^*(s)$ *is a solution of equation* (3.10) *and* $g(s)$ *is a periodic function of* s *of period* 1 *then* $\phi^*(s)\,g(s)$ *is the most general solution.*

So here we find

(3.11) $$\phi(s) = g(s)\ \frac{\Gamma(s+k)\,\Gamma(-s)}{\Gamma(1+k-s)\ \Gamma(s+1)}\ ,$$

where $g(s)$ is a periodic function of s of period 1. Here we suppose that g is a holomorphic function. Now consider the condition $\lim_{|\mathrm{Im}\ s|\to\infty} \phi(s) = 0$.

As $z \to \infty$ in $|\arg\ z| < \pi$ we can use Stirling's formula for the gamma function

$$\Gamma(z) \sim e^{-z}\, z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\ ,$$

hence $\phi(s)$ in (3.11) can then be approximated by

$$\phi(s) \sim \frac{g(s)}{s(1+s)} \left(\frac{s+k}{1-s+k}\right)^k e^2 \left[\frac{(s+k)(1-s+k)}{(-s)(s+1)}\right]^{s-\frac{1}{2}} .$$

Let $s = \gamma + i\sigma$, $0 < \gamma < 1$, $\gamma, \sigma \in \mathbb{R}$. Taking the limit on both sides for $|\sigma| \to \infty$ we conclude that g has to satisfy

$$\frac{g(\gamma+i\sigma)}{(\gamma+i\sigma)(1+\gamma+i\sigma)} \to 0 \quad \text{as } |\sigma| \to \infty .$$

That is

$$g(z) = o(z^2) \quad \text{as } z \to \gamma \pm i\,\infty ,$$

where, since g is periodic , γ can be any real constant. Finally the condition that the constant part of (3.8b) should vanish yields

$$2\pi i \lim_{s\to 0} e^{s(x+i\varepsilon)}(s+1)^2 \,\mathrm{sh}\, \frac{\Gamma(s+1+k)\,\Gamma(-1-s)}{\Gamma(k-s)\,\Gamma(s+2)} \, g(s+1) = -\,2\pi i\delta .$$

The negative sign of the right-hand side is due to the opposite direction of the contour. It is observed that tan $(s+1)h$ is replaced by $(s+1)h$ as has been done in (3.10). We then read

$$\lim_{s\to 0} s\,\Gamma(-s-1)\,g(s) = -\,k ,$$

i.e.

$$\lim_{s\to 0} \frac{s\,g(s)}{\sin(\pi s)} = \frac{-k}{\pi} .$$

Since $g(s)$ is an analytic function on the complex plane it is obvious that $g(0) = -\,k$.

LEMMA 3.2. *If* $g(z)$ *is an analytic function, periodic in* s *with real period* ω, $g(z) = o(z^2)$ *for* $|\mathrm{Im}\, z| \to \infty$, *then* $g(z)$ *is a constant function.*

PROOF. Since $g(z)$ is analytic and $g(z) = o(z^2)$ for $|\mathrm{Im}\, z| \to \infty$, there exists a positive number N so that the following inequality holds

$$|g(z)| \le N\,|1+z|^2$$

for all $z \in \mathbb{C}$. The nth derivative of g(z) at z = s is given by Cauchy's integral formula

$$g^{(n)}(s) = \frac{n!}{2\pi i} \oint \frac{g(z)}{(z-s)^{n+1}}\, dz, \quad n = 1,2,\ldots,$$

Let z_1, z_2 be any two points in the complex plane and let C be a contour such that z_1, z_2 are inside C. Then

$$g'(z_1) - g'(z_2) = \frac{1}{2\pi i} \oint_C \frac{g(z)(z_1-z_2)(2z-z_1-z_2)}{(z-z_1)^2(z-z_2)^2}\, dz$$

or

$$|g'(z_1) - g'(z_2)| \le \frac{|z_1-z_2|}{2\pi} N \oint_C \frac{|1+z|^2 |z-z_1+z-z_2|}{|z-z_1|^2 |z-z_2|^2}\, |dz|$$

Take C to be a circle whose centre is at z_1 and whose radius is $\rho \ge 2|z_1-z_2|$; on C write $z = z_1 + \rho e^{i\theta}$.

Hence $|z-z_1| = \rho$ and $\frac{1}{2}\rho \le |z-z_2| \le \frac{3}{2}\rho$ so that

$$|g'(z_1) - g'(z_2)| \le \frac{|z_1-z_2|}{2\pi} N \int_0^{2\pi} \frac{\{|1+z_1|^2 + 2\rho|1+z_1| + \rho^2\}\frac{5}{2}\rho^2}{\frac{1}{4}\rho^4}\, d\theta$$

Make $\rho \to \infty$ keeping z_1 and z_2 fixed then it is obvious that

$$|g'(z_1) - g'(z_2)| \le \frac{|z_1-z_2|}{2\pi} N \int_0^{2\pi} 10\, d\theta = |z_1-z_2|\, 10\, N.$$

So $\frac{g'(z_1)-g'(z_2)}{z_1-z_2}$ is bounded hence $g''(z) = \lim_{z \to z_2} \frac{g'(z)-g'(z_2)}{z-z_2}$ is bounded for all $z \in \mathbb{C}$. Now apply Liouville's theorem and find that g''(z) must be a constant function. But then, since g(z) is a periodic function, it must be a constant function as well. □

This lemma vields

$$g(s) = -k$$

for all $s \in \mathbb{C}$. Expression $p(x,0)$ can now be solved explicitly. To do so we first integrate $\int e^{s(x+i\varepsilon)}\phi(s)\,ds$. Since the integral should converge a division of the x-axis is made. Define

$$\int_{c-i\infty}^{c+i\infty} -k.e^{\,s(x+i\varepsilon)} \frac{\Gamma(s+k)\,\Gamma(-s)}{\Gamma(1+k-s)\,\Gamma(s+1)}\,ds = \begin{cases} \bar{A}(x) & \text{for } x < 0\,, \\ \bar{B}(x) & \text{for } x > 0\,. \end{cases}$$

Again use is made of the residue theorem

$$\bar{A}(x) = 2\pi i k \sum_{n=1}^{\infty} \lim_{s\to n} (s-n) \frac{\Gamma(s+k)\Gamma(-s)\, e^{s(x+i\varepsilon)}}{\Gamma(1+k-s)\Gamma(s+1)}\,,$$

$$\bar{B}(x) = -\,2\pi i k \sum_{n=0}^{\infty} \lim_{s\to -k-n} (s+k+n) \frac{\Gamma(s+k)\,\Gamma(-s)}{\Gamma(1+k-s)\,\Gamma(s+1)}\, e^{s(x+i\varepsilon)}$$

$$+\, 2\pi i \lim_{s\to 0} s \frac{\Gamma(s+k)\,\Gamma(-s)}{\Gamma(1+k-s)\,\Gamma(s+1)}\, e^{s(x+i\varepsilon)}\,.$$

That is

(3.12a) $$\bar{A}(x) = 2\pi i\,[\,{}_2F_1(k,-k,1,e^{x+i\varepsilon}) - 1\,]\,, \quad x<0\,,$$

(3.12b) $$\bar{B}(x) = -\,2ki\, e^{-k(x+i\varepsilon)} \sin(\pi k) \frac{\Gamma^2(k)}{\Gamma(2k+1)}\, {}_2F_1(k,k,2k+1;e^{-x-i\varepsilon}) + 2\pi i\,, \quad x>0\,.$$

Here ${}_2F_1$ denotes the hypergeometric function. Hence

(3.13a) $$p(x,0) = 2 - {}_2F_1(k,-k;1;e^{x+i\varepsilon})\,, \quad x<0,$$

(3.13b) $$p(x,0) = \frac{e^{-k(x+i\varepsilon)}\,\Gamma(k)}{\Gamma(2k+1)\Gamma(-k)}\, {}_2F_1(k,k;2k+1;\, e^{-x-i\varepsilon})\,, \quad x>0\,.$$

Since $\lim_{x\to-\infty} {}_2F_1(k,k;2k+1;e^{-x-i\varepsilon}) = \lim_{x\to\infty} {}_2F_1(k,k;2k+1;e^{-x-i\varepsilon}) = 1$ we indeed find that

$$\lim_{x\to-\infty} p(x,0) = 1 \quad \text{and} \quad \lim_{x\to\infty} p(x,0) = 0 \, .$$

Moreover it is observed that $p(x,0)$ is bounded for all $x \in \mathbb{R}$ and discontinuous at $x = 0$. Recall that the found expressions (3.13) are approximations for $p(x,0)$, since we replaced (3.9) by (3.10).

As has been stated in section 2 the curve (3.13) moves to the left for increasing ω. Furthermore ε ($\varepsilon < 0$) increases if ω increases. But then $|p(0,0)|$ decreases. This is a well known effect, see e.g. LESSER & BERKLEY [5] .

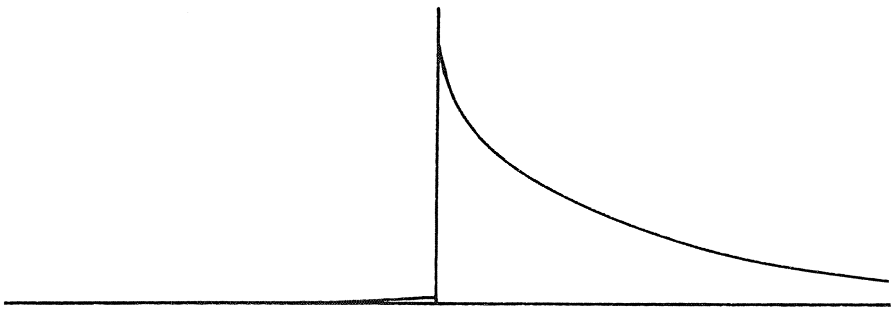


Fig. 3.1. Global behaviour of $|p(x,0)|$

Finally we wish to obtain some asymptotic expression of $p(x,0)$ if h is small, i.e. $|k|$ is large (with small negative imaginary part in the inner ear case). First we treat the case $x > 0$. Set $w = e^{-x-i\varepsilon}$. We make use of the integral representation of the hypergeometric function and find for (3.13b)

$$p(x,0) = \frac{w^k}{\Gamma(-k)\,\Gamma(k+1)} \int_0^1 t^{k-1}(1-t)^k(1-tw)^{-k}\, dt \, ,$$

or

$$\text{(3.14)} \qquad p(x,0) = -\, w^k \frac{\sin \pi k}{k} \int_0^1 \frac{e}{t}^{k\{\ln t + \ln(1-t) - \ln(1-tw)\}} dt \, .$$

We apply the saddle point method for the integral of (3.14). Set $\phi(t) = \ln t + \ln(1-t) - \ln(1-tw)$. Notice that $d\,\phi(t)/dt = 0$ yields $t_{1,2} = \frac{1 \pm \sqrt{1-w}}{w}$. The point of interest is $t_1 = \frac{1-\sqrt{1-w}}{w}$.

Now we define

$$(3.15)\qquad \phi(t) = -\tfrac{1}{2}u^2 + A,$$

where $t = t_1$ if $u = 0$. This implies

$$A = 2\ln\left(\frac{1-\sqrt{1-w}}{w}\right).$$

For (3.14) we read

$$(3.16)\qquad p(x,0) = -w^k \frac{\sin \pi k}{\pi}\left(\frac{1-\sqrt{1-w}}{w}\right)^{2k}\left[\frac{1}{t}\frac{dt}{du}\Big|_{u=0}\right]\int_{-\infty}^{\infty} e^{-\frac{1}{2}ku^2}\,du$$

$$\times\ (1+0|k^{-1}|).$$

Consider the part between brackets. Using (3.15) we find

$$-u\frac{du}{dt} = \frac{d\phi}{dt} = \frac{1-2t+t^2w}{t(1-t)(1-tw)},$$

or, using L'Hospital's rule letting $u \to 0$,

$$\frac{1}{t}\frac{dt}{du}\Big|_{u=0} = \sqrt{\frac{(1-t_1)}{2t_1}}.$$

Hence (3.14) reads

$$(3.17)\qquad p(x,0) \sim -w^k\frac{\sin \pi k}{\pi}\left(\frac{1-\sqrt{1-w}}{w}\right)^{2k}\left(\frac{\sqrt{1-w}}{2(1-\sqrt{1-w})}\right)^{1/2}\left(\frac{2\pi}{k}\right)^{1/2},$$

$$w = e^{-(x+i\varepsilon)},\quad x > 0 .$$

For the case that $x < 0$ we obtain from ERDELYI [3,p.77] the following expression

$$(3.18)\qquad {}_2F_1(k;-k;1;v) = \frac{1}{2\sqrt{\pi}}\left(\frac{1+e^{-\xi}}{1-e^{-\xi}}\right)^{\frac{1}{2}} k^{1/2}\,(e^{k\xi}+ie^{-k\xi})(1+0|k^{-1}|)$$

where ξ is defined by

$$1 - 2v + 2\,(v^2-v)^{1/2} = e^{\xi}\ .$$

Hence for small h (3.13) becomes

(3.19a) $$p(x,0) \sim 2 - \frac{1}{2\sqrt{\pi}} \left(\frac{1-v-(v^2-v)^{1/2}}{v+(v^2-v)^{1/2}} \right)^{1/2} k^{1/2}$$

$$\times \{ (1-2v+2(v^2-v)^{1/2})^k + i(1-2v-2(v^2-v)^{1/2})^k \}$$

$v = e^{x+i\varepsilon}$ and $x < 0$,

(3.19b) $$p(x,0) \sim - w^k \frac{\sin \pi k}{\sqrt{\pi}} \left(\frac{1-\sqrt{1-w}}{w} \right)^{2k} \left(\frac{\sqrt{1-w}}{1-\sqrt{1-w}} \right)^{1/2} k^{-1/2}$$

$w = v^{-1}$ and $x > 0$.

From the above asymptotic approximations it follows that the function p(x,0) is exponentially decreasing when x is positive and it is oscillatory when x is negative. This is in agreement with the known behaviour obtained from other models.

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